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# Computationally Inexpensive Guidance Algorithm for Fuel-Efficient Terminal Descent

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**This article reports on the design of a computationally inexpensive algorithm for the synthesis of control functions for fuel-efficient powered terminal descent of a vehicle. Instead of solving a minimum fuel optimal control problem, the proposed algorithm is based on solving a related optimal control problem for which the solution can be expressed in analytic closed form. This property leads to an algorithm, which, for any initial condition (i.e., in any instance of the problem), will always require a low number of operations to compute the control functions, and no iterations are involved. The fuel performance achieved by the control inputs synthesized by the algorithm is evaluated by numerical simulations of the powered descent phase of a Mars landing and shown to be close to the minimum fuel performance.**

## I. Introduction

IN space exploration adventures involving the landing of an autonomous vehicle on the surface of a given planet, it is often necessary to design algorithms able to synthesize control signals for the powered terminal descent and soft landing of the vehicle. Such algorithms have to synthesize control inputs that will drive the vehicle from a given initial state (position, velocity, and initial fuel mass) to a final landing site with zero velocity. Because the vehicle only carries a limited amount of fuel, it is also a requirement for such algorithms to synthesize control functions that are fuel efficient; ideally, such control functions should yield minimum fuel expenditure. A further important requirement for such algorithms is that they should be computationally inexpensive. This is because the algorithms are expected to allow for computer program implementations able to operate in real time on the onboard computer system. The algorithm used in the Apollo Lunar module[1] dealt with the preceding computational requirement by using low-degree polynomials of time to produce a reference trajectory that passes through the initial state and the final state. In Ref. 2 the algorithm described for a Mars landing operation also relies on the use of low-degree polynomials to produce reference trajectories. A study of the fuel performance of the Apollo guidance algorithm[3] shows that for long horizontal translations (in the range of 2–4 km) the Apollo guidance uses significantly more fuel than the minimum required. Regarding the fuel-efficiency requirement, we previously mentioned that ideally we would like the control function to minimize the fuel consumption. This optimal control problem for (one-dimensional) vertical motions was solved in Ref. 4 using calculus of variations. The same optimal control problem was also solved in Ref. 5 by means of the Pontryagin maximum principle.[6–9] In that work it is proved that (for the particular constraint set considered) this problem is equivalent to the minimum-time (to reach the target) optimal control problem. Also, a switching function that leads to a simple algorithm for control synthesis is constructed.

In the present work, instead of solving a (three-dimensional) minimum fuel optimal control problem, the proposed algorithm is based on solving other related optimal control problems, for which the solutions can be expressed in analytic closed form. This property leads to an algorithm, which, for any initial condition (i.e., in any instance of the problem), will always require a low number of operations to compute the control functions, and no iterations are needed.

The organization of this paper is as follows. In Sec. II we present a description of the dynamical system under consideration, the problem addressed, and the objective of this work. In Sec. III we pose two optimal control problems that are related with a minimum fuel optimal control problem and that are relevant in the context of this work. Sections IV and V are devoted to completely solve these optimal control problems and to derive and present closed-form analytical expressions for their solutions. Section VI is devoted to present a description of the control system (considered in this work) that uses the control functions generated by the proposed algorithm. In Sec. VII we describe the proposed algorithm and discuss the number of elementary operations performed by the algorithm. Results of numerical simulations are presented in Sec. VIII. Conclusions are in Sec. IX.

## II. Description of the Problem and Objective of the Work

It will be assumed that the dynamical system under consideration is described by

$$\dot{x}(t) = Ax(t) + \{1/[m_V + m_F(t)]\}Bu(t) + Bu_g$$
$$\dot{m}_F(t) = -k_F\|u(t)\|_2, \qquad t \in [0, t_f]$$
$$x(0) = x_0 \in \mathbb{R}^6, \qquad m_F(0) = m_{F0} \in \mathbb{R}^+ \tag{1}$$

where

$$A = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{6\times 6}, \quad B = \begin{pmatrix} 0 \\ I \end{pmatrix} \in \mathbb{R}^{6\times 3}, \quad u_g = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix} \in \mathbb{R}^3$$

and $g, m_V, k_F \in \mathbb{R}^+$ are given.

This system models a landing vehicle with dry mass $m_V$, fuel mass $m_F(t)$, and position represented by its Cartesian coordinates $x_1(t), x_2(t), x_3(t)$ [in a stationary frame relative to the landing surface; with $x_3(t)$ being the vertical coordinate]. In this model, the force applied by the thrusters is represented by $u(t)$, $g$ is the gravitational constant of the planet, and $k_F$ is a constant that models the rate of fuel consumption.

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It will be assumed that the control signal is confined to take values on a compact set $U$, where

$$U = \left\{\omega \in \mathbb{R}^3 : \alpha_1 \le \omega_1 \le \beta_1, \alpha_2 \le \omega_2 \le \beta_2, \alpha_3 \le \omega_3 \le \beta_3\right\}$$

$$\alpha_1 = -\alpha(m_V + m_{F0})g, \qquad \beta_1 = \alpha(m_V + m_{F0})g$$

$$\alpha_2 = -\alpha(m_V + m_{F0})g, \qquad \beta_2 = \alpha(m_V + m_{F0})g$$

$$\alpha_3 = \gamma\beta(m_V + m_{F0})g, \qquad \beta_3 = \beta(m_V + m_{F0})g$$

and $\alpha > 0$, $0 \le \gamma < 1$, $\beta > 1$ are given parameters that model the capabilities of the actuators (thrusters). To be more specific, we will assume that

$$u \in \Omega(U, t_f) = \{u \in \mathcal{L}_\infty(0, t_f) : u(t) \in U \text{ a.e. on } [0, t_f]\}$$

Our objective is to design a computationally inexpensive algorithm, which provided with $x_0$ and $m_{F0}$, will find $t_f \in \mathbb{R}^+$ and will synthesize a control function $u \in \Omega(U, t_f)$ for which $x(t_f, x_0, m_{F0}, u) = 0$ and moreover $m_F(t_f, x_0, m_{F0}, u) \ge m_{FL} > 0$ a given lower bound (for the final fuel mass).

We design the algorithm by setting up and solving an appropriately chosen optimal control problem in order to use its solution as the kernel of the algorithm. Notice that, as earlier remarked, ideally we would like to consider and solve the following optimal control problem:

$$\max_{t_f \in \mathbb{R}^+, u \in \Omega(U,t_f), x(t_f, x_0, m_{F0}, u) = 0} m_F(t_f, x_0, m_{F0}, u) \tag{2}$$

In the search for an algorithm having low computational cost, we consider instead two modifications/simplifications of the preceding optimal control problem. In this work, we study and solve two simpler but related optimal control problems and also evaluate the performance of their solutions (in terms of fuel efficiency). The consideration of these much simpler related optimal control problems, which are stated next, is motivated by the fact that their solutions can be expressed in analytic closed form, as we constructively show.

Although our objective, regarding the design of an algorithm, has already been stated, we want to further emphasize that the algorithm to be designed in this work is for the synthesis of control functions satisfying the preceding specified constraints. We remark that the preceding specifications do not include constraints on the path followed by the state of the system. In recent work, reported in Ref. 10, a different approach is taken for the design of an algorithm for powered terminal descent, which is able to take into consideration path constraints.

## III. Two Simple Optimal Control Problems

Let $t_f > 0$ be given. For the dynamical system described by

$$\dot{x}(t) = Ax(t) + [1/(m_V + m_{F0})]Bu(t) + Bu_g$$

$$t \in [0, t_f], \qquad x(0) = x_0 \in \mathbb{R}^6 \tag{3}$$

and the performance functionals,

$$J_1 : \Omega(U, t_f) \longrightarrow \mathbb{R}^+, \qquad J_2 : \Omega(U, t_f) \longrightarrow \mathbb{R}^+$$

defined by,

$$J_1(u, t_f) = \int_0^{t_f} \|u(t)\|_1 \, \mathrm{d}t$$

$$J_2(u, t_f) = \frac{1}{2}\int_0^{t_f} \|u(t)\|_2^2 \, \mathrm{d}t$$

we will consider the following optimal control problems:

$$\min_{u \in \Omega(U,t_f), x(t_f, x_0, u) = 0} J_1(u, t_f) \tag{4}$$

$$\min_{u \in \Omega(U,t_f), x(t_f, x_0, u) = 0} J_2(u, t_f) \tag{5}$$

*Remark 1:* Let us recall that the following inequalities hold:

$$\int_0^{t_f} \|u(t)\|_2 \, \mathrm{d}t \le \sqrt{2t_f}\sqrt{J_2(u, t_f)}$$

$$\int_0^{t_f} \|u(t)\|_2 \, \mathrm{d}t \le J_1(u, t_f)$$

Therefore by solving the optimal control problems (4) and (5), we get solutions that guarantee the best upper bound of the preceding forms for

$$\min_{u \in \Omega(U,t_f), x(t_f, x_0, u) = 0} \int_0^{t_f} \|u(t)\|_2 \, \mathrm{d}t$$

In the next two sections we present a complete treatment for the solutions to the preceding posed optimal control problems that later on will be used in the design of the proposed algorithm. To our knowledge such a treatment, which includes explicit formulas for the solutions, has not appeared in the literature. For the second optimal control problem, (5), with no constraint on the control function (i.e., when $U = \mathbb{R}^3$), the solution is of course well known, and it regularly appears well documented in books covering linear systems theory (e.g., see Refs. 11 and 12). However, when the control is constrained, no explicit formula for its solution is known to be available.

Regarding the existence of solutions for the optimal control problems (4) and (5), we just mention that direct application of existence theorems in optimal control (e.g., see Ref. 7, p. 274, theorem 6) proves the following: if there exists $u \in \Omega(U, t_f)$: $x(t_f, x_0, u) = 0$, then there exist optimal solutions $u^{\mathrm{opt}_1}$, $u^{\mathrm{opt}_2} \in \Omega(U, t_f)$ for problems (4) and (5), respectively.

Because 1) the time $t_f$ is given, 2) the set $U$ is the product of three intervals $U = [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \times [\alpha_3, \beta_3]$, 3) the system (3) is in fact composed of three decoupled systems driven by $u_i$, $i = 1$, 2, 3 (the components of $u$), and 4) the cost functionals can be expressed as the sum of three cost functionals each penalizing each of $u_i$, $i = 1, 2, 3$ (and only one of them), it follows that the optimal control problems (4) and (5) can be solved by equivalently solving three associated independent optimal control subproblems. The next section is devoted to solving the independent optimal control subproblems associated with Eq. (5). The optimal control subproblems associated with Eq. (4) are dealt with in a following section.

## IV. Elementary Optimal Control Problem: Part I

Let $t_f > 0$, $k_1 > 0$, and $k_2 \in \mathbb{R}$ be given. Consider the (scalar) system,

$$\ddot{x}(t) = k_1 u(t) + k_2, \qquad t \in [0, t_f]$$

$$x(0) = x_0 \in \mathbb{R}, \qquad \dot{x}(0) = v_0 \in \mathbb{R} \tag{6}$$

Let $\hat{\alpha}, \hat{\beta} \in \mathbb{R}$ satisfying $\hat{\alpha} < \hat{\beta}$ be given, and define

$$\hat{\Omega}(\hat{\alpha}, \hat{\beta}, t_f) = \{u \in \mathcal{L}_\infty(0, t_f) : \hat{\alpha} \le u(t) \le \hat{\beta} \text{ a.e. on } [0, t_f]\}$$

Define the cost functional $I_2 : \hat{\Omega}(\hat{\alpha}, \hat{\beta}, t_f) \longrightarrow \mathbb{R}^+$ by

$$I_2(u, t_f) = \frac{1}{2}\int_0^{t_f} u^2(t) \, \mathrm{d}t$$

In this section we will consider the following optimal control problem:

$$\min_{u \in \hat{\Omega}(\hat{\alpha}, \hat{\beta}, t_f), x(t_f, x_0, v_0, u) = 0, \dot{x}(t_f, x_0, v_0, u) = 0} I_2(u, t_f) \tag{7}$$

The following notation will be useful in this and following sections:

$$T_R(\hat{\alpha}, \hat{\beta}, x_0, v_0) = \{t_f > 0 : \exists u \in \hat{\Omega}(\hat{\alpha}, \hat{\beta}, t_f),\ \text{satisfying}$$
$$x(t_f, x_0, v_0, u) = 0, \dot{x}(t_f, x_0, v_0, u) = 0\}$$

That is, $T_R(\hat{\alpha}, \hat{\beta}, x_0, v_0)$ is the set of positive final times $t_f$ for which there exists an admissible control that solves the preceding associated reachability problem. We also use here and in the sequel the following standard notation. For given real numbers $r_1 < r_2$, the function $\text{sat}_{[r_1,r_2]} : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by

$$\text{sat}_{[r_1,r_2]}(\xi) = \begin{cases} r_1, & \xi < r_1 \\ \xi, & r_1 \le \xi < r_2 \\ r_2, & r_2 \le \xi \end{cases}$$

Regarding the solution of the preceding optimal control problem, we have the following important results.

*Theorem 1:* Assume that $\hat{\alpha} < -k_2/k_1 < \hat{\beta}$. Let $x_0, v_0 \in \mathbb{R}$ and $t_f > 0$ be given, and assume the following inequality holds:

$$t_f > \inf_{t \in T_R(\hat{\alpha},\hat{\beta},x_0,v_0)} t$$

Under these conditions, the optimal control problem (7) has a unique solution, $u^{opt}$, which can be expressed by

$$u^{opt}(t) = \text{sat}_{[\hat{\alpha},\hat{\beta}]}(at + b), \qquad t \in [0, t_f] \tag{8}$$

where $a, b \in \mathbb{R}$ is the unique solution to the following system of equations:

$$\int_0^{t_f}\int_0^t \text{sat}_{[\hat{\alpha},\hat{\beta}]}(a\tau + b)\,\mathrm{d}\tau\,\mathrm{d}t = \frac{-1}{k_1}\left(x_0 + v_0 t_f + k_2\frac{t_f^2}{2}\right)$$
$$\int_0^{t_f} \text{sat}_{[\hat{\alpha},\hat{\beta}]}(at + b)\,\mathrm{d}t = \frac{-1}{k_1}(v_0 + k_2 t_f) \tag{9}$$

*Remark 2:* Under the assumption that $\hat{\alpha} < -k_2/k_1 < \hat{\beta}$, we make the following important clarifying remarks:

1) Theorem 1 states that if $t_f$ is such that $t_f > \inf_{t \in T_R(\hat{\alpha},\hat{\beta},x_0,v_0)} t$, then the system of equations (9) has an unique solution $(a, b)$ (expressed in analytic closed form in fact 1), which represents via Eq. (8) the unique solution $u^{opt}$ for the optimal control problem (7).

2) It is also straightforward to prove that in case $t_f > 0$ is such that the system of equations (9) has a solution $(a, b)$, then it represents via Eq. (8) the unique solution, $u^{opt}$, for the optimal control problem (7), and (it therefore trivially follows that) $t_f \ge \inf_{t \in T_R(\hat{\alpha},\hat{\beta},x_0,v_0)} t$. Moreover, in case the preceding solution $(a, b)$ is such that $(at + b) \in (\hat{\alpha}, \hat{\beta})$ for some $t \in [0, t_f]$, then it is in fact the unique solution for Eqs. (9), and it also follows that $t_f > \inf_{t \in T_R(\hat{\alpha},\hat{\beta},x_0,v_0)} t$.

*Fact 1:* Under the same assumptions as in theorem 1, the solution $a, b \in \mathbb{R}$ to the system of Eqs. (9) can be expressed in analytic closed form as follows. For notational simplicity we define $x_f = (-1/k_1)[x_0 + v_0 t_f + k_2(t_f^2/2)]$ and $v_f = (-1/k_1)(v_0 + k_2 t_f)$. Then,

*Case 1:* If $a, b$ defined next are such that $\hat{\alpha} \le b \le \hat{\beta}$ and $\hat{\alpha} \le (at_f + b) \le \hat{\beta}$, then they are the solution for Eqs. (9).

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{t_f^3}{6} & \frac{t_f^2}{2} \\ \frac{t_f^2}{2} & t_f \end{pmatrix}^{-1} \begin{pmatrix} x_f \\ v_f \end{pmatrix}$$

*Case 2A:* If $[x_f - \hat{\alpha}(t_f^2/2)] > 0$, $(v_f - \hat{\alpha}t_f) > 0$, and moreover when $a, b$, and $t_{S_1}$ defined next are such that $0 < t_{S_1} < t_f$ and $(at_f + b) \le \hat{\beta}$, then this pair $a, b$ is the solution for Eqs. (9).

$$a = \frac{2}{9}\frac{(v_f - \hat{\alpha}t_f)^3}{\left[x_f - \hat{\alpha}\left(t_f^2/2\right)\right]^2}, \qquad t_{S_1} = t_f - \sqrt{\frac{2(v_f - \hat{\alpha}t_f)}{a}}$$
$$b = \left(\hat{\alpha} - at_{S_1}\right)$$

*Case 2B:* If $[x_f - \hat{\beta}(t_f^2/2)] < 0$, $(v_f - \hat{\beta}t_f) < 0$, and moreover when $a, b$, and $t_{S_1}$ defined next are such that $0 < t_{S_1} < t_f$ and $(at_f + b) \ge \hat{\alpha}$, then this pair $a, b$ is the solution for Eqs. (9).

$$a = \frac{-2}{9}\frac{\left(\hat{\beta}t_f - v_f\right)^3}{\left[\hat{\beta}\left(t_f^2/2\right) - x_f\right]^2}, \qquad t_{S_1} = t_f - \sqrt{\frac{2(\hat{\beta}t_f - v_f)}{|a|}}$$
$$b = \left(\hat{\beta} - at_{S_1}\right)$$

*Case 3A:* If $[x_f - v_f t_f + \hat{\beta}(t_f^2/2)] > 0$, $(v_f - \hat{\beta}t_f) < 0$, and moreover when $a, b$, and $t_{S_2}$ defined next are such that $0 < t_{S_2} < t_f$ and $b \ge \hat{\alpha}$, then this pair $a, b$ is the solution for Eqs. (9).

$$a = \frac{2}{9}\frac{(\hat{\beta}t_f - v_f)^3}{\left[x_f - v_f t_f + \hat{\beta}\left(t_f^2/2\right)\right]^2}, \qquad t_{S_2} = \sqrt{\frac{2(\hat{\beta}t_f - v_f)}{a}}$$
$$b = \left(\hat{\beta} - at_{S_2}\right)$$

*Case 3B:* If $[x_f - v_f t_f + \hat{\alpha}(t_f^2/2)] < 0$, $(v_f - \hat{\alpha}t_f) > 0$, and moreover when $a, b$ and $t_{S_2}$ defined next are such that $0 < t_{S_2} < t_f$ and $b \le \hat{\beta}$, then this pair $a, b$ is the solution for Eqs. (9).

$$a = \frac{-2}{9}\frac{(v_f - \hat{\alpha}t_f)^3}{\left[v_f t_f - \hat{\alpha}\left(t_f^2/2\right) - x_f\right]^2}, \qquad t_{S_2} = \sqrt{\frac{2(v_f - \hat{\alpha}t_f)}{|a|}}$$
$$b = \left(\hat{\alpha} - at_{S_2}\right)$$

*Case 4A:* If $2x_f - v_f t_f + (\hat{\beta}t_f - v_f)t_f - [1/(\hat{\beta} - \hat{\alpha})](\hat{\beta}t_f - v_f)^2 > 0$, $(v_f - \hat{\beta}t_f) < 0$, and moreover when $t_{S_1}, t_{S_2}$, the solutions of

$$t_S^2 - \frac{2(\hat{\beta}t_f - v_f)}{(\hat{\beta} - \hat{\alpha})}t_S + \frac{2}{(\hat{\beta} - \hat{\alpha})}\left[2\frac{(\hat{\beta}t_f - v_f)^2}{(\hat{\beta} - \hat{\alpha})} + 3\left(v_f - \hat{\beta}\frac{t_f}{2}\right)t_f - 3x_f\right] = 0$$

satisfy the condition $0 < t_{S_1} < t_{S_2} < t_f$, then the pair $a, b$ defined next is the solution for Eqs. (9).

$$a = \frac{(\hat{\beta} - \hat{\alpha})}{\left(t_{S_2} - t_{S_1}\right)}, \qquad b = \frac{1}{2}(\hat{\beta} + \hat{\alpha}) - \frac{1}{2}\frac{(\hat{\beta} - \hat{\alpha})\left(t_{S_2} + t_{S_1}\right)}{\left(t_{S_2} - t_{S_1}\right)}$$

*Case 4B:* If $2x_f - v_f t_f + (\hat{\alpha}t_f - v_f)t_f + [1/(\hat{\beta} - \hat{\alpha})](\hat{\alpha}t_f - v_f)^2 < 0$, $(v_f - \hat{\alpha}t_f) > 0$ and moreover when $t_{S_1}, t_{S_2}$, the solutions of

$$t_S^2 + \frac{2(\hat{\alpha}t_f - v_f)}{(\hat{\beta} - \hat{\alpha})}t_S + \frac{2}{(\hat{\beta} - \hat{\alpha})}\left[2\frac{(\hat{\alpha}t_f - v_f)^2}{(\hat{\beta} - \hat{\alpha})} - 3\left(v_f - \hat{\alpha}\frac{t_f}{2}\right)t_f + 3x_f\right] = 0$$

satisfy the condition $0 < t_{S_1} < t_{S_2} < t_f$, then the pair $a, b$ defined next is the solution for Eqs. (9).

$$a = -\frac{(\hat{\beta} - \hat{\alpha})}{\left(t_{S_2} - t_{S_1}\right)}, \qquad b = \frac{1}{2}(\hat{\beta} + \hat{\alpha}) + \frac{1}{2}\frac{(\hat{\beta} - \hat{\alpha})\left(t_{S_2} + t_{S_1}\right)}{\left(t_{S_2} - t_{S_1}\right)}$$

## V. Elementary Optimal Control Problem: Part II

Let $t_f > 0$, $k_1 > 0$, and $k_2 \in \mathbb{R}$ be given. Consider again the (scalar) system

$$\ddot{x}(t) = k_1 u(t) + k_2, \qquad t \in [0, t_f]$$
$$x(0) = x_0 \in \mathbb{R}, \qquad \dot{x}(0) = v_0 \in \mathbb{R} \tag{6}$$

Let $\hat{\alpha}, \hat{\beta} \in \mathbb{R}$ satisfying $\hat{\alpha} < \hat{\beta}$ be given, and denote $\hat{\Omega}(\hat{\alpha}, \hat{\beta}, t_f)$ as before:

$$\hat{\Omega}(\hat{\alpha}, \hat{\beta}, t_f) = \{u \in \mathcal{L}_\infty(0, t_f) : \hat{\alpha} \leq u(t) \leq \hat{\beta} \text{ a.e. on } [0, t_f]\}$$

Define the cost functional $I_1 : \hat{\Omega}(\hat{\alpha}, \hat{\beta}, t_f) \longrightarrow \mathbb{R}^+$ by

$$I_1(u, t_f) = \int_0^{t_f} |u(t)| \, \mathrm{d}t$$

In this section we will consider the following optimal control problem:

$$\min_{u \in \hat{\Omega}(\hat{\alpha}, \hat{\beta}, t_f), x(t_f, x_0, v_0, u) = 0, \dot{x}(t_f, x_0, v_0, u) = 0} I_1(u, t_f) \tag{10}$$

Regarding the solution of the preceding optimal control problem, we have the following important results.

*Fact 2:* Assume that $0 \leq \hat{\alpha} < \hat{\beta}$. Let $x_0, v_0 \in \mathbb{R}$ and $t_f > 0$ be given. Assume there exists $u \in \hat{\Omega}(\hat{\alpha}, \hat{\beta}, t_f)$, such that $x(t_f, x_0, v_0, u) = 0$, $\dot{x}(t_f, x_0, v_0, u) = 0$.

Under these conditions, the optimal control problem (10) has a solution. Moreover, every $u \in \hat{\Omega}(\hat{\alpha}, \hat{\beta}, t_f)$, satisfying $x(t_f, x_0, v_0, u) = 0$ and $\dot{x}(t_f, x_0, v_0, u) = 0$, is in fact optimal. Further, the optimal cost is

$$\min_{u \in \hat{\Omega}(\hat{\alpha}, \hat{\beta}, t_f), x(t_f, x_0, v_0, u) = 0, \dot{x}(t_f, x_0, v_0, u) = 0} I_1(u, t_f) = (1/k_1)(-k_2 t_f - v_0)$$

An analogous statement can be made in case $\hat{\alpha} < \hat{\beta} \leq 0$.

*Theorem 2:* Assume that $0, -k_2/k_1 \in (\hat{\alpha}, \hat{\beta})$. Let $x_0, v_0 \in \mathbb{R}$ and $t_f > 0$ be given, and assume the following inequality holds:

$$t_f > \inf_{t \in T_R(\hat{\alpha}, \hat{\beta}, x_0, v_0)} t$$

Under these conditions, the optimal control problem (10) has a solution. There are only three possible cases that totally characterize the solution of the optimal control problem (10).

1) If there exists $u \in \hat{\Omega}(0, \hat{\beta}, t_f)$, such that $x(t_f, x_0, v_0, u) = 0$ and $\dot{x}(t_f, x_0, v_0, u) = 0$, any such $u$ is an optimal solution for Eq. (10). In this case the optimal cost is

$$\min_{u \in \hat{\Omega}(\hat{\alpha}, \hat{\beta}, t_f), x(t_f, x_0, v_0, u) = 0, \dot{x}(t_f, x_0, v_0, u) = 0} I_1(u, t_f) = (1/k_1)(-k_2 t_f - v_0)$$

2) If there exists $u \in \hat{\Omega}(\hat{\alpha}, 0, t_f)$, such that $x(t_f, x_0, v_0, u) = 0$ and $\dot{x}(t_f, x_0, v_0, u) = 0$, any such $u$ is an optimal solution for Eq. (10). In this case the optimal cost is

$$\min_{u \in \hat{\Omega}(\hat{\alpha}, \hat{\beta}, t_f), x(t_f, x_0, v_0, u) = 0, \dot{x}(t_f, x_0, v_0, u) = 0} I_1(u, t_f) = (1/k_1)(k_2 t_f + v_0)$$

3) If the preceding two constrained reachability problems are infeasible, the optimal control problem Eq. (10) has a unique solution $u^{opt}$. The optimal control is either of the form

$$u^{opt}(t) = \begin{cases} \hat{\alpha}, & 0 \leq t < t_{S_1} \\ 0, & t_{S_1} \leq t < t_{S_2} \\ \hat{\beta}, & t_{S_2} \leq t < t_f \end{cases} \tag{11}$$

or of the form

$$u^{opt}(t) = \begin{cases} \hat{\beta}, & 0 \leq t < t_{S_1} \\ 0, & t_{S_1} \leq t < t_{S_2} \\ \hat{\alpha}, & t_{S_2} \leq t < t_f \end{cases} \tag{12}$$

where the preceding ambiguity and the parameters $0 < t_{S_1} < t_{S_2} < t_f$ are uniquely determined by solving the following system of equations:

$$\int_0^{t_f} \int_0^t u^{opt}(\tau) \, \mathrm{d}\tau \, \mathrm{d}t = \frac{-1}{k_1}\left(x_0 + v_0 t_f + k_2 \frac{t_f^2}{2}\right)$$
$$\int_0^{t_f} u^{opt}(t) \, \mathrm{d}t = \frac{-1}{k_1}(v_0 + k_2 t_f) \tag{13}$$

*Remark 3:* We state here the following observations, which are related to the statement of theorem 2. Assume that $0, -k_2/k_1 \in (\hat{\alpha}, \hat{\beta})$. Let $x_0, v_0 \in \mathbb{R}$ and $t_f > 0$ be given.

1) If there exist functions $u^{(1)}, u^{(2)}$ satisfying the following conditions:

$$u^{(1)} \in \hat{\Omega}(0, \hat{\beta}, t_f), \qquad x\left(t_f, x_0, v_0, u^{(1)}\right) = 0$$
$$\dot{x}\left(t_f, x_0, v_0, u^{(1)}\right) = 0$$

and

$$u^{(2)} \in \hat{\Omega}(\hat{\alpha}, 0, t_f), \qquad x\left(t_f, x_0, v_0, u^{(2)}\right) = 0$$
$$\dot{x}\left(t_f, x_0, v_0, u^{(2)}\right) = 0$$

then, it follows that $u^{(1)} = u^{(2)} = 0$.

2) If there exists a function $u$ such that either $u \in \hat{\Omega}(0, \hat{\beta}, t_f)$ or $u \in \hat{\Omega}(\hat{\alpha}, 0, t_f)$ and it satisfies $x(t_f, x_0, v_0, u) = 0$, $\dot{x}(t_f, x_0, v_0, u) = 0$, [this trivially implies that $t_f \geq \inf_{t \in T_R(\hat{\alpha}, \hat{\beta}, x_0, v_0)} t$], then such a function is an optimal solution for Eq. (10). Moreover, this implies that the system of equations (13) in the unknowns $t_{S_1}, t_{S_2}$, $0 < t_{S_1} < t_{S_2} < t_f$, and the binary variable that determines the expression, expression (11) or expression (12), has no solution. Further, if such a control $u$ is such that $u \neq \hat{\alpha}$ and $u \neq \hat{\beta}$ (i.e., $u$ is not an extremal control as defined in Ref. 7, p. 73), then it follows that $t_f > \inf_{t \in T_R(\hat{\alpha}, \hat{\beta}, x_0, v_0)} t$.

3) If the system of equations (13) in the unknowns $t_{S_1}, t_{S_2}$ ($0 < t_{S_1} < t_{S_2} < t_f$) and the binary variable that determines the expression [expression (11) or expression (12)] has a solution, then it is unique, and this solution totally determines [via Eqs. (11) and (12)] the unique optimal solution for Eq. (10). Further, it follows that $t_f > \inf_{t \in T_R(\hat{\alpha}, \hat{\beta}, x_0, v_0)} t$.

*Fact 3:* Assume that $0, -k_2/k_1 \in (\hat{\alpha}, \hat{\beta})$. Let $x_0, v_0 \in \mathbb{R}$ and $t_f > 0$ be given, and assume the following inequality holds:

$$t_f > \inf_{t \in T_R(\hat{\alpha}, \hat{\beta}, x_0, v_0)} t$$

Assume also that there is not $u \in \hat{\Omega}(\hat{\alpha}, 0, t_f)$, and neither there is $u \in \hat{\Omega}(0, \hat{\beta}, t_f)$, with the property that $x(t_f, x_0, v_0, u) = 0$ and $\dot{x}(t_f, x_0, v_0, u) = 0$.

Then, under these assumptions the unique solution, $u^{opt}$, for the optimal control problem (10) can be expressed in analytic closed form

as follows. As before, we use the definition, $x_f = (-1/k_1)[(x_0 + v_0 t_f + k_2(t_f^2/2)]$ and $v_f = (-1/k_1)(v_0 + k_2 t_f)$.

Then,

*Case A:* If $(t_f - v_f/\hat{\beta})^2 - (1/\hat{\beta} - 1/\hat{\alpha})(v_f^2/\hat{\beta} - 2x_f) \geq 0$, and moreover there exists a solution, $t_{S_1}$, to the equation

$$(\hat{\alpha}^2/2)(1/\hat{\beta} - 1/\hat{\alpha})t_S^2 + \hat{\alpha}(t_f - v_f/\hat{\beta})t_S + \left(v_f^2\big/2\hat{\beta} - x_f\right) = 0$$

with the property that, when we define

$$t_{S_2} = t_f - \left(v_f - \hat{\alpha} t_{S_1}\right)\big/\hat{\beta}$$

the following inequalities hold: $0 < t_{S_1} < t_{S_2} < t_f$: then the optimal control, $u^{opt}$, is given by the expression (11).

*Case B:* If $(t_f - v_f/\hat{\alpha})^2 + (1/\hat{\beta} - 1/\hat{\alpha})(v_f^2/\hat{\alpha} - 2x_f) \geq 0$, and moreover there exists a solution, $t_{S_1}$, to the equation,

$$(-\hat{\beta}^2/2)(1/\hat{\beta} - 1/\hat{\alpha})t_S^2 + \hat{\beta}(t_f - v_f/\hat{\alpha})t_S + \left(v_f^2\big/2\hat{\alpha} - x_f\right) = 0$$

with the property that, when we define

$$t_{S_2} = t_f - \left(v_f - \hat{\beta} t_{S_1}\right) / \hat{\alpha}$$

the following inequalities hold: $0 < t_{S_1} < t_{S_2} < t_f$; then the optimal control, $u^{opt}$, is given by the expression (12).

## VI. Controller for the Original Dynamical System

We recall that the dynamical system under consideration is in fact described by Eq. (1), and optimal control functions $u^{opt_1}$, $u^{opt_2} \in \Omega(U, t_f)$ were found for problems (4) and (5), which involve the dynamical system described by Eq. (3) (instead). Let us denote by $u^{opt}_{x_{C_0}, t_f} \in \Omega(U, t_f)$, an optimal control associated with (4) or (5), where we have made explicit that this optimal control function was found based on the knowledge that the initial condition for the system is $x_{C_0}$ and the time for reaching the target is $t_f$. We define the function $u^*_{x_{C_0}, t_f}$ in the following manner:

$$u^*_{x_{C_0}, t_f}(t) = \begin{cases} u^{opt}_{x_{C_0}, t_f}(t), & 0 \le t \le t_f \\ -(m_V + m_{F0}) u_g, & t > t_f \end{cases} \tag{14}$$

We will consider here the following controller:

$$\dot{x}_C(t) = A x_C(t) + \frac{1}{(m_V + m_{F0})} B u^*_{x_{C_0}, t_f}(t) + B u_g$$

$$t \in \mathbb{R}^+, \quad x_C(0) = x_{C_0}$$

$$u(t) = \frac{[m_V + m_F(t)]}{(m_V + m_{F0})} \left\{ u^*_{x_{C_0}, t_f}(t) - K[x(t) - x_C(t)] \right\} \tag{15}$$

where $K \in \mathbb{R}^{3\times 6}$ is a design parameter to be fixed.

With slightly more generality than Eq. (1), we will assume here that the dynamical system under consideration is described by

$$\dot{x}(t) = A x(t) + \{1/[m_V + m_F(t)]\} B[u(t) + u_d(t)] + B u_g$$

$$\dot{m_F}(t) = -k_F \|u(t)\|_2, \qquad t \in \mathbb{R}^+, \quad x(0) = x_0$$

$$m_F(0) = m_{F0} \in \mathbb{R}^+ \tag{16}$$

where $u_d$ is an input disturbance.

Next we present some straightforward observations regarding the closed-loop system (16) and (15).

*Fact 4:* Let $t_f > 0$ be given, and let $u^{opt}_{x_{C_0}, t_f} \in \Omega(U, t_f)$ be an optimal control for the problem (4) or (5). For the closed-loop system described by (16) and (15), where $u^*_{x_{C_0}, t_f}$ is given by Eq. (14), the following hold:

1) The closed-loop system can be equivalently described by

$$\dot{e}(t) = \left[ A - \frac{1}{(m_V + m_{F0})} B K \right] e(t) + B \left\{ \frac{1}{[m_V + m_F(t)]} u_d(t) \right\}$$

$$e(0) = x_0 - x_{C_0}$$

$$\dot{x_C}(t) = A x_C(t) + \frac{1}{(m_V + m_{F0})} B u^*_{x_{C_0}, t_f}(t) + B u_g, \quad x_C(0) = x_{C_0}$$

$$u(t) = \frac{[m_V + m_F(t)]}{(m_V + m_{F0})} \left[ -K e(t) + u^*_{x_{C_0}, t_f}(t) \right]$$

$$\dot{m_F}(t) = -k_F \|u(t)\|_2, \qquad m_F(0) = m_{F0} \in \mathbb{R}^+ \tag{17}$$

2) If $u_d = 0$ and $x_{C_0} = x_0$, the controller will drive the (state of the) system to the origin at time $t_f$. Moreover,

$$m_F(t_f) = (m_V + m_{F0}) \left\{ \exp\left[ \frac{-k_F \int_0^{t_f} \left\| u^{opt}_{x_{C_0}, t_f}(t) \right\|_2 \mathrm{d}t}{(m_V + m_{F0})} \right] - 1 \right\} + m_{F0}$$

3) If $u_d = 0$, $x_{C_0} = x_0$, and assuming that

$$\exp\left\{ \frac{-k_F \int_0^t \left\| u^{opt}_{x_{C_0}, t_f}(\tau) \right\|_2 \mathrm{d}\tau}{(m_V + m_{F0})} \right\} \left[ u^{opt}_{x_{C_0}, t_f}(t) \right]_3 \ge \alpha_3, \ \forall t \in [0, t_f]$$

it then follows that $u_{[0, t_f]} \in \Omega(U, t_f)$.

Next, we make some comments regarding the design parameter $K$. We want to use $K$ in order to reduce the effects of $x_0 \ne x_{C_0}$ (i.e., the exact initial condition of the dynamical system is unknown) and the effects of $u_d \ne 0$ (i.e., there are input disturbances that model for instance, the effects of wind, imperfect response of the actuators, etc). It is therefore appropriate to look at the dynamics of the error, $e$, which obeys the following equation:

$$\dot{e}(t) = [A - 1/(m_V + m_{F0}) B K] e(t) + B\{1/[m_V + m_F(t)] u_d(t)\}$$

$$e(0) = x_0 - x_{C_0}$$

The preceding equation is, of course, coupled with the dynamic equation of the fuel mass $m_F$ (which we have not rewritten). Hence, our design objectives can be recast in terms of the preceding linear differential equation in the following manner:

1) $K$ must ensure that the trivial solution of the preceding unforced system will be exponentially stable (i.e., the matrix $\{A - [1/(m_V + m_{F0})] B K\}$ must be Hurwitz).

2) $K$ must provide for good attenuation of the input response, or equivalently $K$ must be chosen such that it reduces some appropriate system norm.

Notice that the first condition can be satisfied, for instance, by choosing $K = (K_1 \ \ K_2)$; $K_1, K_2 \in \mathbb{R}^{3\times 3}$; $K_1^T = K_1 > 0$; and $K_2^T = K_2 > 0$. [The stability in this case can be proved by using a quadratic Lyapunov function represented by the matrix $P = \begin{pmatrix} kK_1 & I \\ I & kI \end{pmatrix}$ with $k > 0$ and $K_1 > (1/k^2) I$.]

## VII. Algorithm Description

We present here a description of the proposed algorithm for the synthesis of control functions, which is based on either one of the two optimal control problems we have solved. In this description, which is presented in a pseudocode, other algorithms are also called. The algorithm Optimal_Control_Problem is the most important one as it constitutes the kernel of the proposed algorithm, and moreover it is the central topic of this paper. It uses the appropriate closed-form analytical expression (developed in Secs. IV and V) in order to return a solution of the optimal control problem under consideration (where $t_f$ is fixed and it is passed as an argument). It returns *status*, which can be either *feasible* or *infeasible*, and also returns *parameters*, which is a set of parameters that totally describe the optimal control. For instance, if the optimal control problem under consideration is Eq. (5), these parameters are $a_1, b_1, a_2, b_2, a_3, b_3$. The algorithm Final_Fuel_Mass_Evaluation computes and returns $m_F(t_f)$ or an approximation for $m_F(t_f)$. The function of the algorithm Find_Maximum does not require explanation.

*Algorithm 1 (Basic_Algorithm 1).*

$t_f \longleftarrow t_{f_{\text{UPPER}}}$;
$i \longleftarrow 1$;
(parameters, status) $\longleftarrow$ Optimal_Control_Problem($m_V, m_{F0}, g, t_f, \alpha, \beta, \gamma, x_0$);
**While** status = feasible,
  Parameters($i$) $\longleftarrow$ parameters;
  $T_f(i) \longleftarrow t_f$;
  $M_F(i) \longleftarrow$ Final_Fuel_Mass_Evaluation($m_V, m_{F0}, g, k_F, t_f, \alpha, \beta, \gamma$, parameters);
  $t_f \longleftarrow t_f - t_{f_{\text{STEP}}}$;
  $i \longleftarrow i + 1$;
  (parameters, status) $\longleftarrow$ Optimal_Control_Problem($m_V, m_{F0}, g, t_f, \alpha, \beta, \gamma, x_0$);
**End**
$(m_F, k) \longleftarrow$ Find_Maximum ($M_F$);
$t_f \longleftarrow T_f(k)$;
parameters $\longleftarrow$ Parameters($k$);

The number of elementary operations needed in the proposed algorithm strongly depends on the following components:

1) $N_{\text{OCP}}$ is the number of elementary operations needed to compute a solution for the optimal control problem under consideration. That is, the number of operations to evaluate the formula in fact 1 for each of the components of $u$ in case optimal control problem (5) is being considered, or the formula in fact 3 for each of the components of $u$ in case optimal control problem (4) is used.

2) $N_{\text{FFME}}[T_f(i)]$ is the number of elementary operations needed to compute (an approximation for) $m_F[T_f(i)]$. (See discussion in Remark 4.)

3) $N$ is the final value for $i$ after leaving the **While** loop.

Therefore, a figure or estimate of the number of elementary operations required in the algorithm is given by the expression

$$N N_{\text{OCP}} + \sum_{i=1}^{N-1} N_{\text{FFME}}[T_f(i)]$$

For instance, in case $t_{f_{\text{UPPER}}} = 120$, $t_{f_{\text{STEP}}} = 1$, and $N = 62$ (i.e., for $t_f = 60$ the optimal control problem is feasible, but for $t_f = 59$ it is infeasible), the preceding expression becomes

$$62\, N_{\text{OCP}} + \sum_{j=0}^{60} N_{\text{FFME}}(120 - j)$$

*Remark 4:* Regarding the algorithm Final_Fuel_Mass_Evaluation, it is important to remark that $m_F(t_f)$ can be computed by means of an explicit formula. Recall that

$$m_F(t_f) = (m_V + m_{F0})\left\{\exp\left[\frac{-k_F \int_0^{t_f} \|u^{opt}(t)\|_2\, dt}{(m_V + m_{F0})}\right] - 1\right\} + m_{F0}$$

Now, we assume first that $u^{opt}$ is the solution for the optimal control problem (5). Then, it is easy to see (from the solution of the preceding optimal control problem) that

$$\int_0^{t_f} \|u^{opt}(t)\|_2\, dt$$

can always be expressed in the following manner:

$$\int_0^{t_f} \|u^{opt}(t)\|_2\, dt = \sum_j \int_{t_j}^{t_{j+1}} \sqrt{\left[u_1^{opt}(t)\right]^2 + \left[u_2^{opt}(t)\right]^2 + \left[u_3^{opt}(t)\right]^2}\, dt$$

where the intervals $[t_j, t_{j+1}]$ that form a partition of $[0, t_f]$ can be constructed (from $u^{opt}$) with the property that

$$\sqrt{\left[u_1^{opt}(t)\right]^2 + \left[u_2^{opt}(t)\right]^2 + \left[u_3^{opt}(t)\right]^2} = \sqrt{a_{4j}t^2 + b_{4j}t + c_{4j}} \qquad \forall t \in [t_j, t_{j+1}]$$

where (there are only two possibilities) either $a_{4j} = b_{4j} = 0$, or $a_{4j} > 0$ and $\Delta_j = (4a_{4j}c_{4j} - b_{4j}^2) \geq 0$.

The preceding constants $a_{4j}, b_{4j}, c_{4j}$ can always be computed from the parameterization that defines $u^{opt}$. (The number of intervals is not bigger than seven, and the number could be as small as one.) Now, because

$$\int_{t_j}^{t_{j+1}} \sqrt{a_{4j}t^2 + b_{4j}t + c_{4j}}\, dt$$

can be computed by means of an explicit formula (e.g., see Ref. 13), our claim follows. In case $a_{4j} > 0$, $\Delta_j = (4a_{4j}c_{4j} - b_{4j}^2) = 0$,

$$\int_{t_j}^{t_{j+1}} \sqrt{a_{4j}t^2 + b_{4j}t + c_{4j}}\, dt = \frac{1}{2}\sqrt{a_{4j}}\left\{\left|t + \frac{b_{4j}}{2a_{4j}}\right|\left(t + \frac{b_{4j}}{2a_{4j}}\right)\right\}\Bigg|_{t_j}^{t_{j+1}}$$

In case $a_{4j} > 0$, $\Delta_j = (4a_{4j}c_{4j} - b_{4j}^2) > 0$,

$$\int_{t_j}^{t_{j+1}} \sqrt{a_{4j}t^2 + b_{4j}t + c_{4j}}\, dt = \frac{1}{2}\left\{\left(t + \frac{b_{4j}}{2a_{4j}}\right)\sqrt{a_{4j}t^2 + b_{4j}t + c_{4j}} + \frac{\Delta_j}{4a_{4j}^{3/2}}\ell n\left[\sqrt{a_{4j}}\left(t + \frac{b_{4j}}{2a_{4j}}\right) + \sqrt{a_{4j}t^2 + b_{4j}t + c_{4j}}\right]\right\}\Bigg|_{t_j}^{t_{j+1}}$$

In the case in which $u^{opt}$ is a solution for the optimal control problem (4), it follows that by adopting a simple convention for the selection of an optimal control for the cases in which the solution to the optimal control problem (4) is not unique the preceding discussion [regarding an explicit formula to compute $\int_0^{t_f} \|u^{opt}(t)\|_2\, dt$] also applies here. The selection convention we adopt is as follows (refer to theorem 2). Regarding the elementary optimal control problem (10), the following selection rule is used:

1) In case there exists $u \in \hat{\Omega}(0, \hat{\beta}, t_f)$, such that $x(t_f, x_0, v_0, u) = 0$ and $\dot{x}(t_f, x_0, v_0, u) = 0$, select $u \in \hat{\Omega}(0, \hat{\beta}, t_f)$ [: $x(t_f, x_0, v_0, u) = 0$ and $\dot{x}(t_f, x_0, v_0, u) = 0$] such that it minimizes $I_2(u, t_f)$.

2) In case there exists $u \in \hat{\Omega}(\hat{\alpha}, 0, t_f)$, such that $x(t_f, x_0, v_0, u) = 0$ and $\dot{x}(t_f, x_0, v_0, u) = 0$, select $u \in \hat{\Omega}(\hat{\alpha}, 0, t_f)$ [: $x(t_f, x_0, v_0, u) = 0$ and $\dot{x}(t_f, x_0, v_0, u) = 0$] such that it minimizes $I_2(u, t_f)$.

The algorithm Optimal_Control_Problem uses the preceding selection convention.

A computationally less expensive variation of the preceding algorithm, Basic_Algorithm 1, is presented next. This algorithm just applies a bisection method (using $t_{f_{\text{UPPER}}}$, $t_{f_{\text{LOWER}}}$, and $t_{f_{\text{RESOLUTION}}}$) in order to compute a solution of the optimal control problem, under consideration, at the minimum found value of $t_f$ for which it is feasible.

*Algorithm 2 (Basic_Algorithm 2).*

**While** $(t_{f_{\text{UPPER}}} - t_{f_{\text{LOWER}}}) > t_{f_{\text{RESOLUTION}}}$,
  $t_f \longleftarrow (t_{f_{\text{UPPER}}} - t_{f_{\text{LOWER}}})/2$;
  (parameters_temp, status) $\longleftarrow$ Optimal_Control_Problem ($m_V$, $m_{F0}, g, t_f, \alpha, \beta, \gamma, x_0$);
  **If** status = feasible,
  $t_{f_{\text{UPPER}}} \longleftarrow t_f$;
  parameters $\longleftarrow$ parameters_temp;
  **Else**
  $t_{f_{\text{LOWER}}} \longleftarrow t_f$;
  **End**
**End**
$t_f \longleftarrow t_{f_{\text{UPPER}}}$;

*Remark 5:* Two different methods can be used in order to select an adequate value for $t_{f_{\text{UPPER}}}$ (used in the proposed algorithm): a theoretical method and a practical one. The theoretical method is based on the fact that the distance from the constraint set $U$ to the origin is positive. In other words, after the thrusters are turned on they cannot go off, which implies that there is a minimum of fuel expenditure per unit time. Because the vehicle only has $m_{F0}$ amount of fuel mass, we are able to compute a theoretical upper bound for $t_f$. Because

$$m_F(0) - m_F(t_f) = k_F \int_0^{t_f} \|u(t)\|_2\, dt$$

and

$$\min_{u \in \Omega(U, t_f)} \|u(t)\|_2 = \alpha_3 = \gamma\beta(m_V + m_{F0})g$$

then we have that

$$t_f \leq \frac{m_{F0}}{k_F\gamma\beta(m_V + m_{F0})g}$$

Therefore, based on that, we can select $t_{f_{\text{UPPER}}}$ as follows:

$$t_{f_{\text{UPPER}}} = \frac{m_{F0}}{k_F\gamma\beta(m_V + m_{F0})g}$$

Notice that the preceding theoretical upper bound for $t_f$ could be very conservative. Therefore, the use of this upper bound (as the value for $t_{f\text{UPPER}}$) could negatively impact on the computational cost of the algorithm. Using the same data as in the numerical simulations (see Sec. VIII), we have that the theoretical upper bound for $t_f$ is $m_{F0}/k_F\gamma\beta(m_V + m_{F0})g = 157.93$ [s]. A practical method, based on the results obtained from running a large number of numerical simulations (using different initial conditions), could provide a less conservative value for $t_{f\text{UPPER}}$. In this method, a safety margin is added to the largest of the values for $t_f$ obtained when running that large number of numerical simulations, and this is the value to be selected as $t_{f\text{UPPER}}$. In our specific case concerning the vehicle whose data are given in Sec. VIII, we have adopted this practical method. The value thus selected was $t_{f\text{UPPER}} = 120$ [s].

## VIII. Numerical Simulations

To evaluate the fuel efficiency that the proposed algorithm achieves, we performed numerical simulations and compared the final fuel mass with the final fuel mass for the minimum fuel optimal control problem (2). In the results presented next we used first Basic_Algorithm 1 based on the solution for the optimal control problem (5) (see Table 1), and later we used Basic_Algorithm 1 based on a solution for the optimal control problem (4) (see Table 2). The values of the parameters that described the dynamical system in all of the optimal control problems considered in these numerical simulations are the following:

$$m_V = 1505 \text{ [kg]}, \qquad m_{F0} = 400 \text{ [kg]}, \qquad g = 3.73 \text{ [m/s}^2\text{]}$$

$$k_F = 0.0005084138 \text{ [kg/N} \cdot \text{s]}$$

The domain $U$, used when computing the optimal value for the optimal control problem (2), is described by

$$\alpha = 0.7, \qquad \beta = 1.71, \qquad \gamma = 0.41$$

**Table 1 Results of Basic_Algorithm 1 based on Eq. (5) vs the optimal value for Eq. (2)**

| $x_0^T$ | $t_f^*$, s | $m_F^*(t_f^*)$, kg | $t_f$, s | $m_F(t_f)$, kg | $\frac{[m_F^*(t_f^*) - m_F(t_f)]}{[m_{F0} - m_F^*(t_f^*)]}100$ |
|---|---|---|---|---|---|
| (2000, 0, 3000, 50, 0, −20) | 78.11 | 71.05 | 82 | 63.00 | 2.45 |
| (2000, 0, 3000, 50, 50, −20) | 77.91 | 56.80 | 82 | 51.61 | 1.51 |
| (2000, 0, 3000, 0, 50, −20) | 69.82 | 111.01 | 76 | 101.64 | 3.24 |
| (2000, 0, 3000, −50, 50, −20) | 69.83 | 120.48 | 76 | 107.94 | 4.49 |
| (2000, 0, 3000, −50, 0, −20) | 69.89 | 142.71 | 76 | 124.02 | 7.26 |
| (2000, 0, 3000, 50, 0, −40) | 80.33 | 58.87 | 82 | 51.03 | 2.30 |
| (2000, 0, 3000, 50, 50, −40) | 77.61 | 46.85 | 82 | 40.59 | 1.77 |
| (2000, 0, 3000, 0, 50, −40) | 58.41 | 122.17 | 63 | 116.11 | 2.18 |
| (2000, 0, 3000, −50, 50, −40) | 58.50 | 136.61 | 63 | 128.25 | 3.17 |
| (2000, 0, 3000, −50, 0, −40) | 58.72 | 160.59 | 63 | 147.49 | 5.47 |
| (2000, 0, 3000, 50, 0, −60) | 78.17 | 46.55 | 82 | 38.18 | 2.37 |
| (2000, 0, 3000, 50, 50, −60) | 79.61 | 37.15 | 82 | 28.85 | 2.29 |
| (2000, 0, 3000, 0, 50, −60) | 54.44 | 116.83 | 57 | 112.02 | 1.70 |
| (2000, 0, 3000, −50, 50, −60) | 52.29 | 140.65 | 56 | 131.69 | 3.45 |
| (2000, 0, 3000, −50, 0, −60) | 51.47 | 163.10 | 56 | 151.90 | 4.73 |
| (2000, 0, 3000, 50, 0, −80) | 76.82 | 33.50 | 82 | 24.58 | 2.43 |
| (2000, 0, 3000, 50, 50, −80) | 76.75 | 21.48 | 82 | 16.16 | 1.41 |
| (2000, 0, 3000, 0, 50, −80) | 54.07 | 106.57 | 56 | 101.38 | 1.77 |
| (2000, 0, 3000, −50, 50, −80) | 46.67 | 136.48 | 51 | 129.66 | 2.59 |
| (2000, 0, 3000, −50, 0, −80) | 47.00 | 159.76 | 51 | 149.15 | 4.42 |

**Table 2 Results of Basic_Algorithm 1 based on Eq. (4) vs the optimal value for Eq. (2)**

| $x_0^T$ | $t_f^*$, s | $m_F^*(t_f^*)$, kg | $t_f$, s | $m_F(t_f)$, kg | $\frac{[m_F^*(t_f^*) - m_F(t_f)]}{[m_{F0} - m_F^*(t_f^*)]}100$ |
|---|---|---|---|---|---|
| (2000, 0, 3000, 50, 0, −20) | 78.11 | 71.05 | 82 | 61.96 | 2.76 |
| (2000, 0, 3000, 50, 50, −20) | 77.91 | 56.80 | 82 | 46.60 | 2.97 |
| (2000, 0, 3000, 0, 50, −20) | 69.82 | 111.01 | 76 | 94.00 | 5.89 |
| (2000, 0, 3000, −50, 50, −20) | 69.83 | 120.48 | 76 | 103.50 | 6.07 |
| (2000, 0, 3000, −50, 0, −20) | 69.89 | 142.71 | 76 | 124.00 | 7.27 |
| (2000, 0, 3000, 50, 0, −40) | 80.33 | 58.87 | 82 | 50.05 | 2.59 |
| (2000, 0, 3000, 50, 50, −40) | 77.61 | 46.85 | 82 | 35.79 | 3.13 |
| (2000, 0, 3000, 0, 50, −40) | 58.41 | 122.17 | 63 | 111.18 | 3.96 |
| (2000, 0, 3000, −50, 50, −40) | 58.50 | 136.61 | 63 | 125.21 | 4.33 |
| (2000, 0, 3000, −50, 0, −40) | 58.72 | 160.59 | 63 | 147.43 | 5.50 |
| (2000, 0, 3000, 50, 0, −60) | 78.17 | 46.55 | 82 | 37.24 | 2.63 |
| (2000, 0, 3000, 50, 50, −60) | 79.61 | 37.15 | 82 | 24.47 | 3.49 |
| (2000, 0, 3000, 0, 50, −60) | 54.44 | 116.83 | 56 | 109.10 | 2.73 |
| (2000, 0, 3000, −50, 50, −60) | 52.29 | 140.65 | 56 | 129.64 | 4.25 |
| (2000, 0, 3000, −50, 0, −60) | 51.47 | 163.10 | 56 | 151.87 | 4.74 |
| (2000, 0, 3000, 50, 0, −80) | 76.82 | 33.50 | 82 | 23.67 | 2.68 |
| (2000, 0, 3000, 50, 50, −80) | 76.75 | 21.48 | 82 | 12.15 | 2.46 |
| (2000, 0, 3000, 0, 50, −80) | 54.07 | 106.57 | 56 | 99.19 | 2.52 |
| (2000, 0, 3000, −50, 50, −80) | 46.67 | 136.48 | 51 | 128.41 | 3.06 |
| (2000, 0, 3000, −50, 0, −80) | 47.00 | 159.76 | 51 | 148.97 | 4.49 |

For all of the results generated by Basic_Algorithm 1, the (more conservative) value of $\gamma = 0.44$ was used. This is a conservative measure in order to ensure that the control functions $u$ generated by Eq. (15) will satisfy the constraint $u_{[0,t_f]} \in \Omega(U, t_f)$ for the preceding defined set $U$. Also, we are using the following assumption for all of the numerical simulations: $u_d = 0$, $x_{C_0} = x_0$. In Tables 1 and 2 the first column $x_0^T$ corresponds to initial conditions, the third column $m_F^*(t_f^*)$ corresponds to the final fuel mass for the minimum fuel optimal control problem (2), and the fifth column $m_F(t_f)$ corresponds to the final fuel mass achieved by a control synthesized by the proposed algorithm. [It can be noticed, from the tables, that the control functions synthesized by Basic_Algorithm 1 based on Eq. (5) achieve better fuel performance than the ones synthesized by Basic_Algorithm 1 based on Eq. (4).]

With the intention of providing an illustration, we present, in Figs. 1–3, the graphical representation of the control function synthesized by Basic_Algorithm 1 based on Eq. (5) for the initial condition $x_0^T = (2000, 0, 3000, 0, 50, -60)$. Figures 4–6 show the corresponding control function generated by the controller (15) (with $x_{C_0} = x_0$). The motion of the system [with initial condition $x_0^T = (2000, 0, 3000, 0, 50, -60)$, and $u_d = 0$] in response to that control function is illustrated by Figs. 7–12.

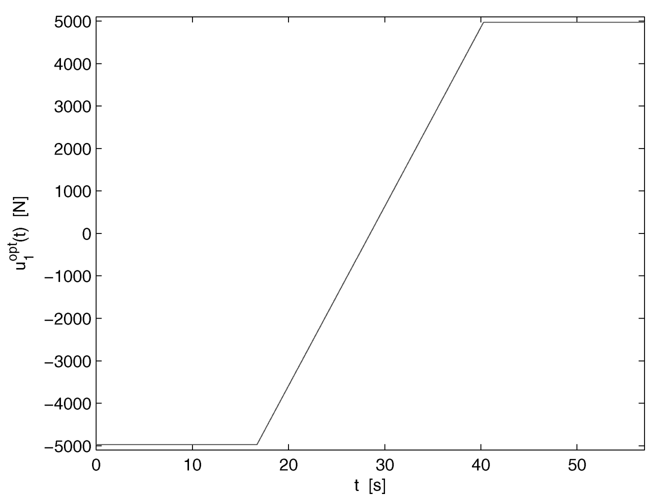


**Fig. 1 First component of the control function $u^{opt}$ synthesized by Basic_Algorithm 1 based on Eq. (5) for the initial condition $x_0^T$ = (2000, 0, 3000, 0, 50, −60).**

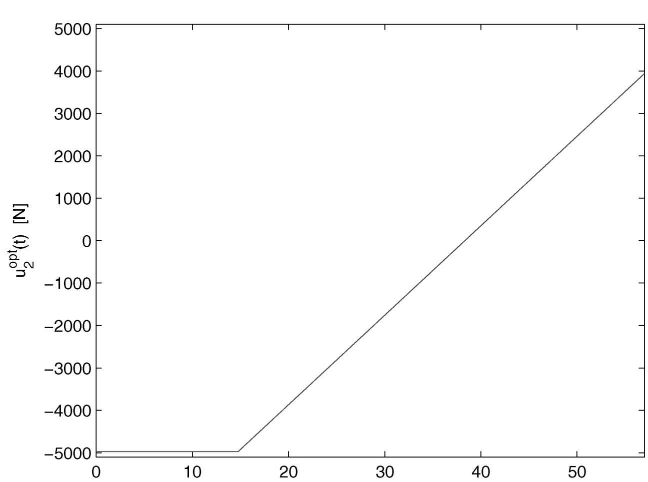


**Fig. 2 Second component of the control function $u^{opt}$ synthesized by Basic_Algorithm 1 based on Eq. (5) for the initial condition $x_0^T$ = (2000, 0, 3000, 0, 50, −60).**

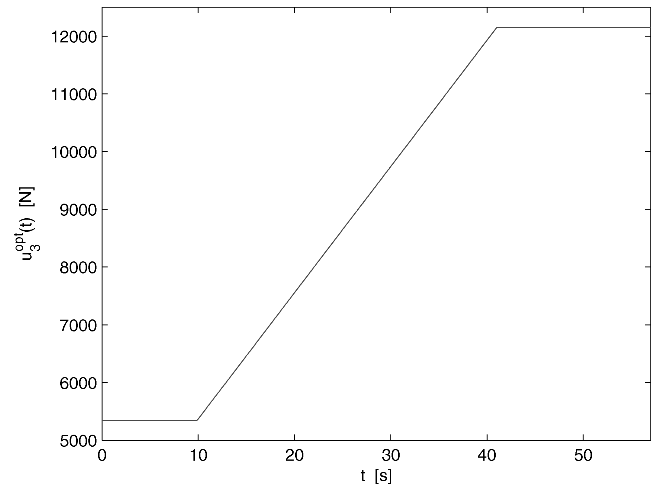


**Fig. 3 Third component of the control function $u^{opt}$ synthesized by Basic_Algorithm 1 based on Eq. (5) for the initial condition $x_0^T$ = (2000, 0, 3000, 0, 50, −60).**

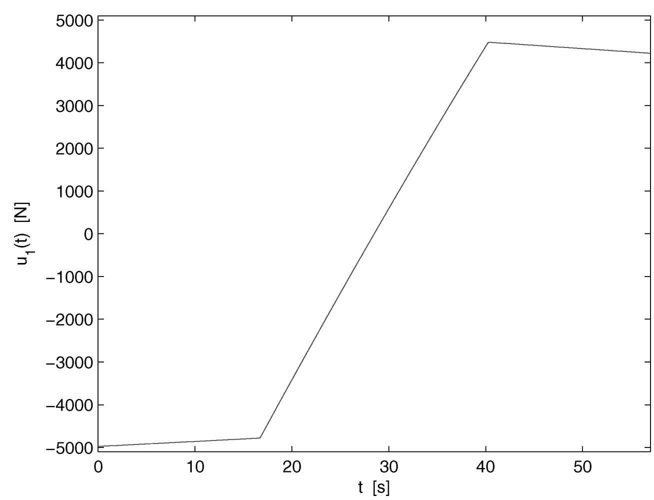


**Fig. 4 First component of the control function $u$ generated by the controller (15) in response to $u^{opt}$ shown in Figs. 1–3.**

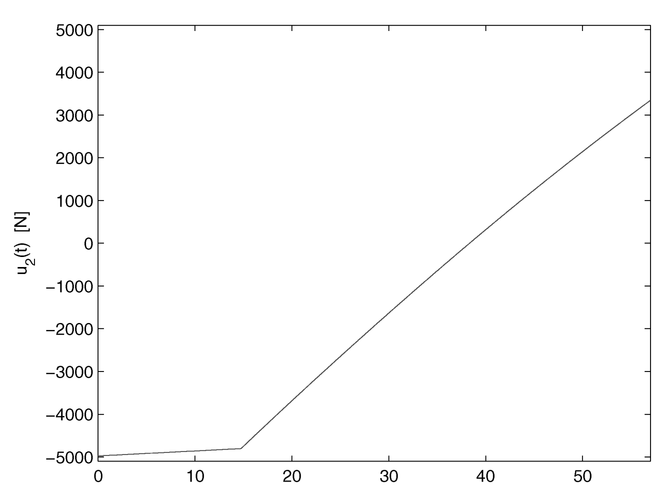


**Fig. 5 Second component of the control function $u$ generated by the controller (15) in response to $u^{opt}$ shown in Figs. 1–3.**

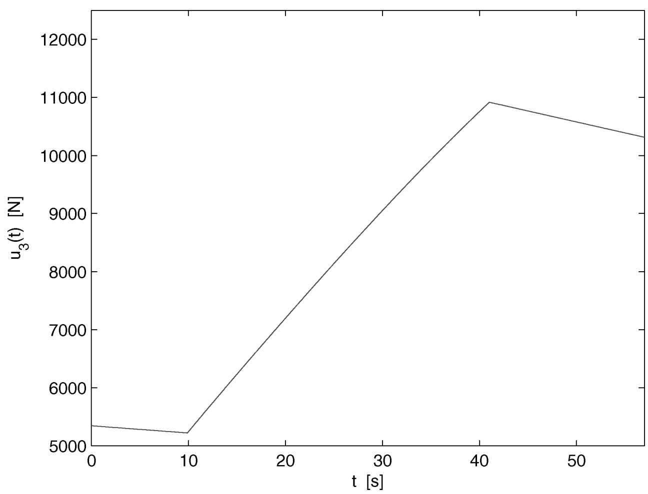


**Fig. 6 Third component of the control function $u$ generated by the controller (15) in response to $u^{opt}$ shown in Figs. 1–3.**

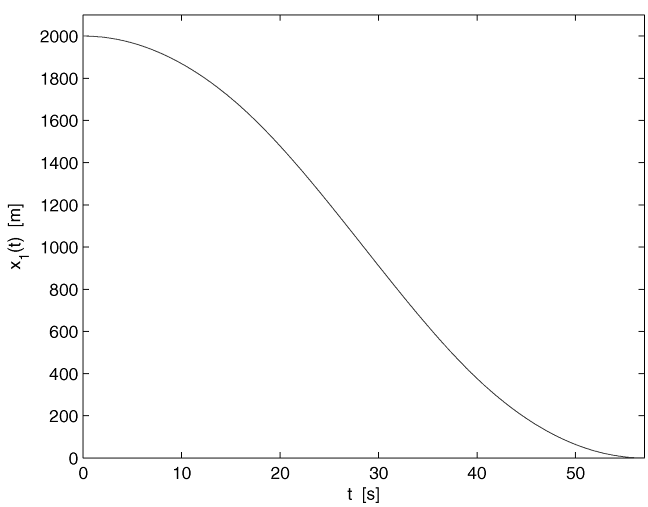


**Fig. 7 Response of the system $x_1$ to the control function $u$ shown in Figs. 4–6 [for $x_0^T = (2000, 0, 3000, 0, 50, -60)$].**

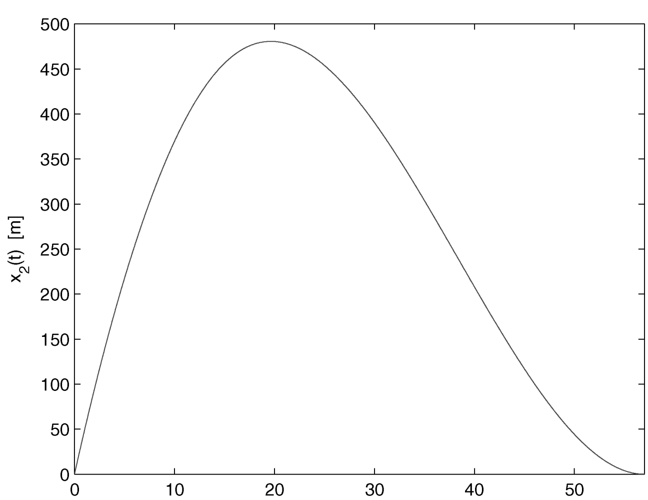


**Fig. 8 Response of the system $x_2$ to the control function $u$ shown in Figs. 4–6 [for $x_0^T = (2000, 0, 3000, 0, 50, -60)$].**

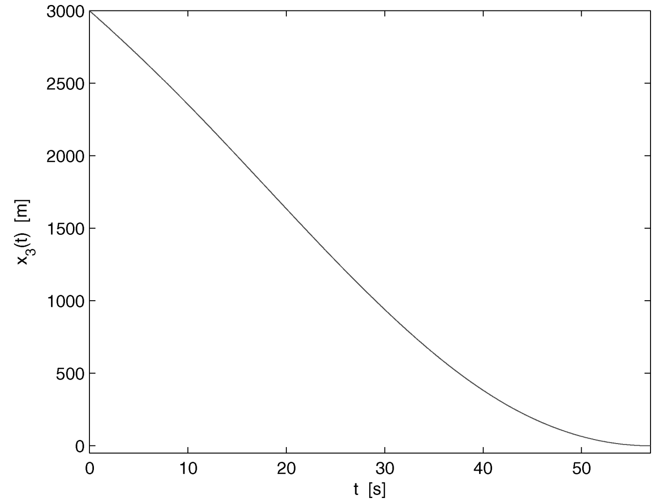


**Fig. 9 Response of the system $x_3$ to the control function $u$ shown in Figs. 4–6 [for $x_0^T = (2000, 0, 3000, 0, 50, -60)$].**

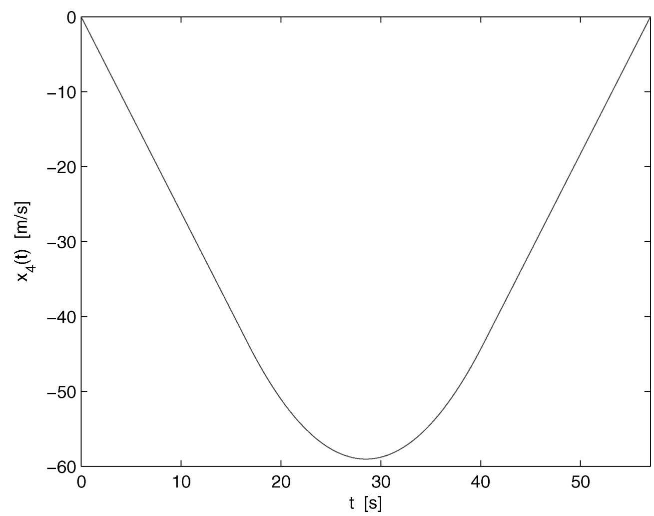


**Fig. 10 Response of the system $x_4$ to the control function $u$ shown in Figs. 4–6 [for $x_0^T = (2000, 0, 3000, 0, 50, -60)$].**

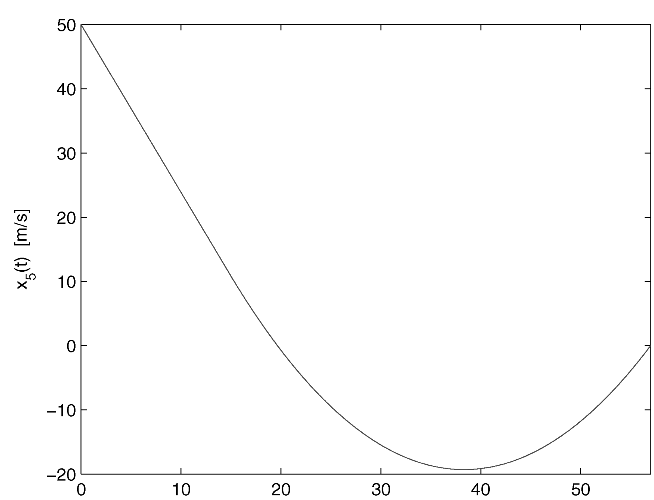


**Fig. 11 Response of the system $x_5$ to the control function $u$ shown in Figs. 4–6 [for $x_0^T = (2000, 0, 3000, 0, 50, -60)$].**

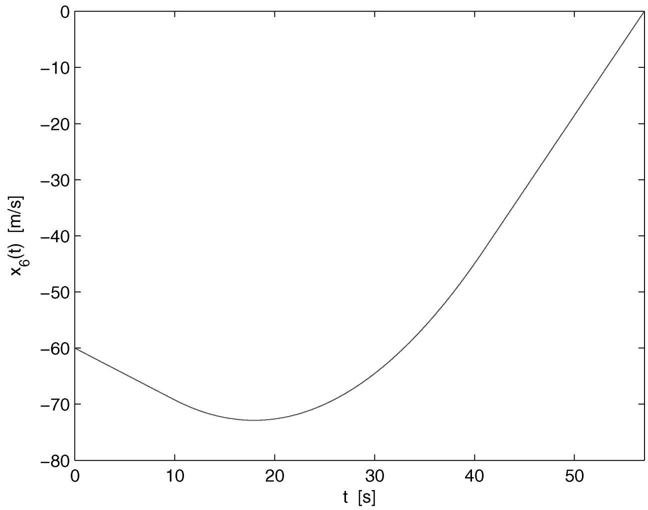


**Fig. 12 Response of the system $x_6$ to the control function $u$ shown in Figs. 4–6 [for $x_0^T = (2000, 0, 3000, 0, 50, -60)$].**

## IX. Conclusions

We developed and proposed a computationally inexpensive algorithm for the synthesis of control functions for fuel-efficient powered terminal descent of a vehicle. Instead of solving a minimum fuel optimal control problem, the proposed algorithm is based on explicit formulas for the solutions of other related optimal control problems. Two of these related optimal control problems were solved, and explicit closed-form analytical expressions for their solutions were derived and presented. These explicit formulas constitute the kernel of the proposed algorithm. Because the algorithm is based on one of these formulas, it possesses the following important properties. It has low computational cost. It is reliable, that is, it requires no iteration in synthesizing a control function, and moreover when a solution is physically possible (for a related dynamical system) it will always provide a control function.

To evaluate the fuel efficiency achieved by the proposed algorithm, numerical simulations were performed. The results obtained by means of the proposed algorithm were compared with the optimal values of an associated minimum fuel optimal control problem (which achieves the best possible fuel efficiency). The results of these numerical simulations show that the control inputs synthesized by the proposed algorithm achieve close to minimum fuel performance. The control inputs synthesized by the algorithm are therefore referred to as fuel-efficient controls.

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